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LETTER TO THE EDITOR

On the most probable path for diffusion processes

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Abstract. The Lagrangian which determines the most probable path for diffusion processes is obtained by a simple and intuitive technique based entirely on the use of path integrals.

We shall discuss here the problem of the most probable path for diffusion processes. We have solved this problem in one dimension (Langouche *et al* 1978a) in order to be able to obtain a steepest descent approximation for the probability density. We shall show here that we can extend the method used there to the case of N slow variables q^{μ}, $\mu = 1, 2, \ldots N$, where the diffusion matrix is such that there is no curvature. We use a path integral approach which is simple and intuitive, and which avoids unnecessary complications. The case with curvature will be treated in a forthcoming paper. The Fokker–Planck equation of our system is

$$\dot{P}(\boldsymbol{q}, t) = \frac{\partial}{\partial q^{\mu}}\left(A^{\mu}(\boldsymbol{q}) + \frac{1}{2}\frac{\partial}{\partial q^{\nu}} D^{\mu\nu}(\boldsymbol{q})\right) P(\boldsymbol{q}, t) \tag{1}$$

where $A^{\mu}(\boldsymbol{q})$ is the drift and $D^{\mu\nu}(\boldsymbol{q})$ the diffusion matrix, which we use as the contravariant metric tensor in the curvilinear coordinates q^{μ} (Graham 1977, Langouche *et al* 1978c). The fundamental solution $P(\boldsymbol{Q}, t; \boldsymbol{Q}_0, t_0)$ of (1) such that $P(\boldsymbol{Q}, t_0; \boldsymbol{Q}_0, t_0) = \delta(\boldsymbol{Q} - \boldsymbol{Q}_0)$ admits the functional integral representations

$$P(\boldsymbol{Q}, t; \boldsymbol{Q}_0, t_0) = \int_{\gamma\; \boldsymbol{q}(t_0)=\boldsymbol{Q}_0}^{\boldsymbol{q}(t)=\boldsymbol{Q}} \mathscr{D}\left(\frac{\boldsymbol{q}(\tau)}{(D(\boldsymbol{q}))^{1/2}}\right) \exp\left[-\int_{t_0}^{t} \mathrm{d}\tau\, L^{\gamma}(\dot{q}^{\sigma}, A^{\sigma}(\boldsymbol{q}), D^{\rho\sigma}(\boldsymbol{q}))\right] \tag{2}$$

where $D(\boldsymbol{q}) = \det D^{\mu\nu}(\boldsymbol{q})$ and γ stands for the discretisation involved in the definition of the path integral as we have shown in (Langouche *et al* 1978c). We recall that L^{γ}, which we call the Lagrangian, depends on the discretisation and that consequently it is not uniquely determined. The interpretation of (2) as a sum over all paths such that $\boldsymbol{q}(t_0) = \boldsymbol{Q}_0$, $\boldsymbol{q}(t) = \boldsymbol{Q}$, is well known, and also its difficulties since the paths are non differentiable ($\dot{q}$ does not exist as a derivative in (2)) and the representation not unique. Nevertheless this interpretation strongly suggests the possibility of giving a meaning to the most probable trajectory between $\boldsymbol{Q}_0$ and $\boldsymbol{Q}$. It should be clear from the beginning that it only makes sense to speak of the probability of a small region in q-space around a path, and that our work must consist in comparing the probabilities of these regions, and this also tells us that we can restrict ourselves to consider regions around differentiable paths. In fact we would be especially interested in determining a Lagrangian $L_p(q^{\sigma}, \dot{q}^{\sigma})$ which we shall call here the Onsager–Machlup function, such that its Euler–Lagrange equations determine, when integrated with the boundary conditions $\boldsymbol{q}(t_0)=\boldsymbol{Q}_0$, $\boldsymbol{q}(t)=\boldsymbol{Q}$, the most probable path. (The paths should be then twice differentiable.)

Our method in Langouche *et al* (1978a) consisted in transforming the functional integral representation (2) to another one which is independent of the discretisation (in probabilistic language this means eliminating the stochastic integrals in the integrand of (2)), then a unique Lagrangian $L(q^\sigma, \dot{q}^\sigma)$ is determined by (2) and it is the solution to our problem.

In N degrees of freedom and when there is no curvature we can treat our problem in rectilinear coordinates (i.e. constant diffusion). We recall that the contravariant drift vector $h^\mu(q)$ associated with (1) is

$$h^\mu(\boldsymbol{q})=A^\mu(\boldsymbol{q})+\frac{1}{2}(D(\boldsymbol{q}))^{1/2}\,\partial_\nu\left(\frac{D^{\mu\nu}(\boldsymbol{q})}{(D(\boldsymbol{q}))^{1/2}}\right). \tag{3}$$

The technique of Bach *et al* (1978) can be generalised in a straightforward way when $h_\mu(\boldsymbol{q})$ is a gradient $\partial_\mu\,\phi(\boldsymbol{q})$. Let us suppose then that we are in orthogonal rectilinear coordinates where $D_{\mu\nu}(\boldsymbol{q})=\delta_{\mu\nu}$, $D(\boldsymbol{q})=1$, the representation can be taken as the direct generalisation of formula (32) in Langouche *et al* (1978c) and is

$$P(\boldsymbol{Q},t;\boldsymbol{Q}_0,t_0)=\int_{\substack{\gamma(\alpha)\\ q(t_0)=\boldsymbol{Q}_0}}^{q(t)=\boldsymbol{Q}}\mathscr{D}\boldsymbol{q}(\tau)\exp\left[-\int_{t_0}^{t}\mathrm{d}\tau\sum_{\mu=1}^{N}\left(\tfrac{1}{2}(\dot{q}^\mu+A^\mu(\boldsymbol{q}))^2-\alpha\,\partial_\mu A^\mu(\boldsymbol{q})\right)\right] \tag{4}$$

where the discretisation $\gamma(\alpha)$ is either the $\gamma_1(\alpha)$ of Leschke and Schmutz (1977) or our $\gamma_2(\alpha)$ which are equivalent here (Langouche *et al* 1978b, c). If we are in the case when $A^\mu(\boldsymbol{q})=A_\mu(\boldsymbol{q})=\partial_\mu\,\phi(\boldsymbol{q})$ we can integrate by parts the term $\dot{q}^\mu A^\mu(\boldsymbol{q})$ in the exponential of (4), as we have shown in the appendix B of Langouche *et al* (1978b), using the formula

$$\sum_{\mu=1}^{N}\int_{t_0}^{t}\mathrm{d}\tau\,\dot{q}^\mu A^\mu(\boldsymbol{q}(\tau))=\phi(\boldsymbol{q}(t))-\phi(\boldsymbol{q}(t_0))-(\tfrac{1}{2}-\alpha)\sum_{\mu=1}^{N}\int_{t_0}^{t}\mathrm{d}\tau\,\partial_\mu A^\mu(\boldsymbol{q}(\tau)). \tag{5}$$

Doing this we obtain from (4) that

$$P(\boldsymbol{Q},t;\boldsymbol{Q}_0,t_0)=\exp[\phi(\boldsymbol{Q}_0)-\phi(\boldsymbol{q}(t)]\int_{\boldsymbol{q}(t_0)=\boldsymbol{Q}_0}^{\boldsymbol{q}(t)=\boldsymbol{Q}}\mathscr{D}\boldsymbol{q}(\tau)\exp\left[-\int_{t_0}^{t}\mathrm{d}\tau\left[\tfrac{1}{2}\sum_{\mu=1}^{N}(\dot{q}^\mu)^2+V(\boldsymbol{q})\right]\right] \tag{6}$$

where $V(\boldsymbol{q})=\frac{1}{2}\sum_{\mu=1}^{N}[A^\mu(\boldsymbol{q})^2-\partial_\mu A^\mu(\boldsymbol{q})^2-\partial_\mu A^\mu(\boldsymbol{q})]$, and the path integral is now independent of the discretisation. This independence can either be proved directly as in Langouche *et al* (1978a) or one can remark that the corresponding "Hamiltonian" in the operator formalism has no ordering ambiguities (Langouche *et al* 1977, Leschke and Schmutz 1977). The condition $A_\mu=\partial_\mu\phi$ is realised for instance if the Graham–Haken (Graham 1973, Haken 1975) potential conditions are satisfied and if the nondissipative part $\overset{0}{A_\mu}$ of A_μ vanishes, in this case $2A_\mu=\partial_\mu\tilde{\phi}$ when the stationary probability is $p_s(\boldsymbol{q})=Z^{-1}\exp(-\tilde{\phi})$, and consequently $\tilde{\phi}=2\phi$ (see Enz (1977), formulae (4.4) and (4.11)). This property has also been used in a different context by Garrido *et al* (1978). The most probable path can now be read from (6); it will be the trajectory that makes the action in (6) stationary and L_p can be taken, when we add to the Lagrangian in (6) the total derivative $\dot{\phi}(\boldsymbol{q}(\tau))=\sum\dot{q}^\mu A^\mu(\boldsymbol{q}(\tau))$, as the Lagrangian in (4) for $\alpha=\frac{1}{2}$. (The reason for this is of course that the extra term in (5) vanishes for $\alpha=\frac{1}{2}$, i.e. when (5) is a Stratonovic stochastic integral.) We have been working in cartesian coordinates, but as L_p is a scalar (the most probable path cannot depend on the coordinate system), we can immediately write down the expression $\bar{L}_p(\bar{q}^\sigma, \dot{\bar{q}}^\sigma)$ in general curvilinear coordinates $\bar{q}^\mu = \bar{q}^\mu(\boldsymbol{q})$ as

$$\bar{L}_\mu(\bar{q}^\sigma, \dot{\bar{q}}^\sigma) = \tfrac{1}{2}\bar{D}_{\mu\nu}(\bar{q})(\dot{\bar{q}}^\mu - h^\mu(\bar{q}))(\dot{\bar{q}}^\nu - h^\nu(\bar{q})) - \frac{1}{2}(\bar{D}(\bar{q}))^{1/2}\,\partial_\mu\left(\frac{h^\mu(\bar{q})}{(\bar{D}(\bar{q}))^{1/2}}\right) \tag{7}$$

with

$$\bar{D}^{\mu\nu} = \frac{\partial \bar{q}^\mu}{\partial q^\alpha}\frac{\partial \bar{q}^\nu}{\partial q^\beta}\,\delta^{\alpha\beta}$$

and

$$h^\mu(\bar{q}) = \bar{A}(\bar{q}) + \frac{1}{2}(\bar{D}(\bar{q}))^{1/2}\,\partial_\nu\left(\frac{\bar{D}^{\mu\nu}(\bar{q})}{(\bar{D}(\bar{q}))^{1/2}}\right) \tag{8}$$

$$\bar{A}^\mu(\bar{q}) = \frac{\partial \bar{q}^\mu}{\partial q^\nu}A^\nu(q) - \frac{1}{2}\,\delta^{\nu\lambda}\frac{\partial^2 \bar{q}^\mu}{\partial q^\nu\,\partial q^\lambda}. \tag{9}$$

One can check immediately that (7) is a scalar which reduces to the Lagrangian in (4) for $\alpha = \frac{1}{2}$ when one is in cartesian coordinates. Let us remark that for one degree of freedom, $N = 1$, there is no curvature and the drift is always a gradient and consequently the problem is completely solved by the previous considerations (Langouche *et al* 1978a). But now let us justify our claim that the unique Lagrangian free of discretisation ambiguities that one obtains is indeed the solution to our problem.

This is easy to do starting from (6). The probability of the paths in a small region of volume characterised by a small parameter η around the differentiable curve $\boldsymbol{y} = \boldsymbol{y}(t)$ will be ($t_j = t_0 + j\epsilon$, $t - t_0 = (n+1)\epsilon$, $\boldsymbol{y}(t_0) = \boldsymbol{y}_0 = \boldsymbol{q}_0$ fixed):

$$P[\boldsymbol{y}(t), \eta] = \lim_{n\to\infty}\int_{\Gamma_\eta}\prod_{i=1}^{n+1}\prod_{\mu=1}^{N}\mathrm{d}q_i^\mu\prod_{j=1}^{n+1}P(\boldsymbol{q}_j, t_j; \boldsymbol{q}_{j-1}, t_{j-1}) \tag{10}$$

where we have put $y_i \equiv y(t_i)$ and the integration is in the region Γ_η such that $y_i^\mu - \eta \leq q_i^\mu \leq y_i^\mu + \eta$. From (6) discretising as in Langouche *et al* (1978a), one obtains

$$P(\boldsymbol{q}_j, t_j; \boldsymbol{q}_{j-1}, t_{j-1}) = \exp[\phi(\boldsymbol{q}_{j-1}) - \phi(\boldsymbol{q}_j)]\frac{1}{(2\pi\epsilon)^{N/2}}\exp\left(-\sum_{\mu=1}^{N}\frac{(\bar{\Delta}_j^\mu)^2}{2\epsilon} - \epsilon V(\boldsymbol{q}_{j-1})\right) \tag{11}$$

where $\bar{\Delta}_j^\mu \equiv q_j^\mu - q_{j-1}^\mu$. We now substitute in (10) after performing the change of variable $\boldsymbol{q}_i = \boldsymbol{y}_i + \boldsymbol{x}_i$. We have ($\Delta_j^\mu \equiv x_j^\mu - x_{j-1}^\mu$)

$$\begin{aligned} P[\boldsymbol{y}(t), \eta] = {} & \exp\left[-\epsilon\sum_{j=1}^{m+1}\left(\sum_{\mu=1}^{N}\frac{(\dot{y}_{j-1}^\mu)^2}{2\epsilon} + V(\boldsymbol{y}_{j-1})\right)\right]\times\int_{\Gamma'_\eta}\prod_{i=1}^{n+1}\prod_{\mu=1}^{N}\left(\frac{\mathrm{d}x_i^\mu}{(2\pi\epsilon)^{1/2}}\right) \\ & \times\exp[\phi(\boldsymbol{y}_0) - \phi(\boldsymbol{y}_{n+1} + \boldsymbol{x}_{n+1})]\exp\sum_{j=1}^{N}\left[-\sum_{\mu=1}^{N}\left(\frac{(\Delta_j^\mu)^2}{2\epsilon} + \dot{y}_{j-1}^\mu\,\Delta_j^\mu\right)\right. \\ & \left. -\epsilon(V(\boldsymbol{y}_{j-1} + \boldsymbol{x}_{j-1}) - V(\boldsymbol{y}_{j-1}))\right] \end{aligned} \tag{12}$$

with Γ'_η such that $-\eta \leq x_i^\mu \leq \eta$ and where we have used $\bar{\Delta}^\mu = \epsilon\dot{y}_{j-1}^\mu + \Delta_j^\mu + \mathrm{O}(\epsilon^2)$ since the curve $\boldsymbol{y}(t)$ is differentiable. (Note that $\Delta_j^\mu = 0(\epsilon^{1/2})$ and that in (12) we have omitted terms $\mathrm{O}(\epsilon^{3/2})$ and higher since they do not contribute to the functional integral (we do not write explicitly from now on the limit $n\to\infty$ that has to be understood in (12)). We recall that we are only interested in (12) when $\eta \to 0$ since the quantity of interest is the quotient $P[y(t), \eta]/P[z(t), \eta)]$ around two differentiable curves $y(t)$ and $z(t)$ when $\eta \to 0$. This implies that $\phi(y_{n+1}+x_{n+1})$ in (16) can be replaced by $\phi(y(t))$ since $|x_{n+1}| = O(\eta)$. Consider now for any function $f^\mu(t)$ the sum $\Sigma_{j,\mu} f^\mu(t_{j-1})\, \Delta_j^\mu$. One has (putting $f^\mu(t_{j-1}) = f_{j-1}^\mu$ and noting that $x_0^\mu = 0$):

$$\sum_{j=1}^{n+1} \sum_{\mu=1}^{N} f_{j-1}^\mu (x_j^\mu - x_{j-1}^\mu) = \sum_{\mu=1}^{N} f_{n+1}^\mu x_{n+1}^\mu + \sum_{j=1}^{n+1} \sum_{\mu=1}^{N} x_j^\mu (f_j^\mu - f_{j-1}^\mu). \tag{13}$$

The first term in the right hand side $\Sigma_\mu f^\mu(t) x_{n+1}^\mu = O(\eta)$ where $n \to \infty$, and the second one has the value

$$\sum_{j=1}^{n+1} \sum_{\mu=1}^{N} x_j^\mu \left(\epsilon \frac{\partial f^\mu(t_{j-1})}{\partial t_{j-1}} + O(\epsilon^2) \right) = \sum_{j=1}^{n+1} \epsilon O(\eta) = O(\eta) \tag{14}$$

where we have used $\Sigma_j \epsilon = (n+1)\epsilon = t - t_0$. The exponential of a sum like (13) will then only contribute a term $\exp(O(\eta))$ and need not be considered this allows us to get rid of the term $\dot{y}_{j-1}^\mu\, \Delta_j^\mu$ in (12). The last term in (12) is

$$\sum_{j=1}^{n+1} \epsilon (V(y_{j-1} + x_{j-1}) - V(y_{j-1})) = \sum_{j=1}^{n+1} \epsilon O(\eta) = O(\eta) \tag{15}$$

and does not contribute either when $\eta \to 0$. Since $A_\mu = \partial_\mu \phi$ here we have

$$\phi(y(t)) - \phi(y_0) = \int_{t_0}^{t} \mathrm{d}\tau\, \dot{y}^\mu(\tau) A^\mu y(\tau)$$

and (12) gives (taking the limit $n \to \infty$ $(\epsilon \to 0)$ and replacing $V(y(\tau))$ by its value)

$$P[y(t), \eta]_{\eta \to 0} = \exp\left[-\int_{t_0}^{t} \mathrm{d}\tau \sum_{\mu=1}^{N} (\tfrac{1}{2}(\dot{y}^\mu + A^\mu y(\tau)^2 - \tfrac{1}{2}\partial_\mu A^\mu y(\tau)) \right]$$
$$\times \lim_{n \to \infty} \int_{\Gamma'_\eta} \prod_{i=1}^{n+1} \prod_{\mu=1}^{N} \left(\frac{\mathrm{d}x_i^\mu}{(2\pi\epsilon)^{1/2}} \right) \exp\left[-\sum_{j=1}^{n+1} \sum_{\mu=1}^{N} \frac{(x_j^\mu - x_{j-1}^\mu)^2}{2\epsilon} \right]. \tag{16}$$

The second term on the right hand side of (16) is just the Wiener measure $P_w(\Gamma'_\eta)$ of the paths $x(\tau)$, $|x(\tau)| \leq \eta$, in the region Γ'_η around the path $x(\tau) \equiv 0$. P_w is independent of the curve $y(t)$ and consequently this term cancels out in the quotient of the probabilities around two differentiable curves thus showing that

$$R = \lim_{\eta \to 0} \frac{P[y(t), \eta]}{P[z(t), \eta]} = \exp\left[-\int_{t_0}^{t} \mathrm{d}\tau (L_p(y^\sigma, \dot{y}^\sigma) - L_p(z^\sigma, \dot{z}^\sigma)) \right] \tag{17}$$

where $L_p(y^\sigma, \dot{y}^\sigma)$ is the Lagrangian in (4) for $\alpha = \frac{1}{2}$. The most probable path is of course the curve $y(t)$ such that $R > 1$ for any other curve $z(t)$, and (17) shows then that L_p is the Onsager–Machlup function as stated before (in arbitrary curvilinear coordinates L_p is given by (7)). We note that the end point of the path $y(t)$ is free in our derivation, if we fix it then the term $\exp[\phi(y_0) - \phi y(t)]$ will cancel in the quotient (17) and one can use for L_p the Langrangian under the functional integral in (6), this is precisely what we did in our steepest descent calculation in Langouche *et al* (1978a). The result (7) has also been obtained using probabilistics methods by Stratonovic (1971), and Dürr and Bach (1978), for one degree of freedom, and by Ito (1978), in the general case $(A_\mu \neq \partial_\mu \phi)$. Our result for the steepest descent approximation has also been corroborated recently with probabilistic techniques in Bach *et al* (1978).

Let us treat now the general case when A_μ is not a gradient. The probability $P[y(t), \eta]$ is given by (10) but we use now instead of (11) the exact short time propagator up to order ϵ, which is, in rectilinear coordinates (as we are in Euclidean space we go again to these coordinates), the direct generalisation of formula (16) in Dürr and Bach (1978). One has

$$P(q_j, t_j; q_{j-1}, t_{j-1}) = \frac{1}{(2\pi\epsilon)^{N/2}} \exp\left[-\sum_{\mu=1}^{N} \frac{(\bar{\Delta}_j^\mu)^2}{2\epsilon}\right][1 - \bar{\Delta}_j^\mu A^\mu(q_{j-1})$$
$$+\tfrac{1}{2}\epsilon(\partial_\mu A^\mu(q_{j-1}) - A^\mu(q_{j-1})A^\mu(q_{j-1}) - \tfrac{1}{2}\bar{\Delta}_j^\mu \bar{\Delta}_j^\nu(\partial_\mu A^\nu(q_{j-1})$$
$$-A^\mu(q_{j-1})A^\nu(q_{j-1})) + O(\epsilon^{3/2})]. \tag{18}$$

Introducing the mid-point discretisation $q_{j-1}^{\sigma(1/2)} = \frac{1}{2}(q_{j-1}^\sigma + q_j^\sigma)$ expression (18) can be written as ($V(q)$ is as before):

$$\frac{1}{(2\pi\epsilon)^{N/2}} \exp\left[\sum_{\mu=1}^{N}\left(-\frac{(\bar{\Delta}_j^\mu)^2}{2\epsilon} - \bar{\Delta}_j^\mu A^\mu(q_{j-1}^{(1/2)})\right) - \epsilon V(q_{j-1})\right]. \tag{19}$$

We note that (19) could have been obtained directly from (4) in the discretisation $\gamma_1(\frac{1}{2})$. Replacing (19) in (10) and performing as before the change of variable $q_i = y_i + x_i$ one obtains (after dropping terms $O(\epsilon\eta)$ or $O(\epsilon^{3/2})$):

$$P[y(t), \eta] = \exp \sum_{j=1}^{n+1}\left[\sum_{\mu=1}^{N}(-\tfrac{1}{2}\epsilon(\dot{y}_{j-1}^\mu)^2 - \epsilon\dot{y}_{j-1}^\mu A^\mu(y_{j-1})) - \epsilon V(y_{j-1})\right]$$
$$\times \int \prod_{i=1}^{n+1}\prod_{\mu=1}^{N}\left(\frac{dx_i^\mu}{(2\pi\epsilon)^{1/2}}\right) \exp \sum_{j=1}^{n+1}\left[\sum_{\mu=1}^{N}\left(-\frac{(\Delta_j^\mu)^2}{2\epsilon} - \dot{y}_{j-1}^\mu \Delta_j^\mu\right.\right.$$
$$\left.\left. - \Delta_j^\mu A^\mu(y_{j-1}^{(1/2)} + x_{j-1}^{(1/2)}) - \epsilon(V(y_{j-1} + x_{j-1}) - V(x_{j-1})\right)\right]. \tag{20}$$

We repeat now the arguments after (12) (formulae (13) to (15)) to eliminate the same terms as there. We note that $\Delta_j^\mu A^\mu(y_{j-1}^{(1/2)} + x_{j-1}^{(1/2)}) = \Delta_j^\mu A^\mu(y_{j-1} + x_{j-1}^{(1/2)}) + O(\epsilon^{3/2}) - \Delta_j^\mu f^\mu(t_{j-1}, x_{j-1}^{(1/2)})$ since $\Delta_j^\mu = O(\epsilon)^{1/2}$ and we obtain then from (20) taking the limit $n \to \infty$ ($\epsilon \to 0$):

$$P[y(t), \eta]_{\eta\to 0} = \exp\left[-\int_{t_0}^{t} d\tau \sum_{\mu=1}^{N}[\tfrac{1}{2}(\dot{y}^\mu + A^\mu(y(\tau)))^2 - \tfrac{1}{2}\partial_\mu A^\mu(y(\tau))]\right]$$
$$\times \lim_{n\to\infty}\int \prod_{i=1}^{n+1}\prod_{\mu=1}^{N}\left(\frac{dx_i^\mu}{(2\pi\epsilon)^{1/2}}\right) \exp \sum_{j=1}^{n+1}\sum_{\mu=1}^{N}\left[-\frac{(x_j^\mu - x_{j-1}^\mu)^2}{2\epsilon}\right.$$
$$\left. - (x_j^\mu - x_{j-1}^\mu) f^\mu(t_{j-1}, x_{j-1}^{1/2})\right]. \tag{21}$$

The term $\Sigma_{i,\mu}\, \Delta_j^\mu f^\mu(t_{j-1}, x_{j-1}^{(1/2)})$ being discretised in the mid-point is a stochastic integral in the Stratonovic sense and then the second term in the right hand side of (21) is just the expectation value of this stochastic integral in Γ'_η. Dividing this quantity by $P_w(\Gamma'_\eta)$, the Wiener measure of Γ'_η that we have introduced before, we obtain the conditional expectation value which has been shown by probabilistic methods to have the limit 1 when $\eta \to 0$ by Ito (1978, formula (A.3)). We have then

$$P[y(t), \eta] = \exp\left[-\int_{t_0}^{t} d\tau \sum_{\mu=1}^{N}[\tfrac{1}{2}(\bar{y}^\mu + A^\mu(y))^2 - \tfrac{1}{2}\partial_\mu A^\mu(y(t))]\right] . P_W(\Gamma'_\mu) \tag{22}$$

which is again (16) thus showing that the Lagrangian in (9) for $\alpha = \frac{1}{2}$ is the Onsager–Machlup function in the general case (in arbitrary curvilinear coordinates (11)).

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